CC10: Unit-(1+2): Mechanics

Unit-1:

Coplanar forces in general:

1. State the following four fundamental principles of statics:

   (a) The principle of physical independence of forces.
   (b) The Principle of Transmissibility of forces.
   (c) The Principle of Action and Reaction.
   (d) The Principle of parallelogram of forces.

2. Analytically deduce the resultant of two concurrent forces P and Q.

3. Resolved analytically the force R into two component forces P and Q which make angles α and β with the line of action of R.

4. If R be the resultant of two forces act at a point O and any transversal cut the lines of action of the forces P, Q, R at the point A, B, C respectively, show that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$.

5. Derive the resultant of two (a) like parallel forces. (b) unlike unequal parallel forces.

6. Derive the resultant of three coplanar forces.

7. (a) When two forces are in equilibrium? (b) When three coplanar forces are in equilibrium?

8. (a) State and prove Lami's theorem. (b) State and prove converse of Lami's theorem.

9. If three forces acting on a rigid body are in equilibrium, the show that (i) the forces must be coplanar and (ii) the forces either meet at a point, or all of them are parallel.

10. Define (a) moment of a force about a point. (b) moment of a force about an axis.

11. State Varignon's theorem on moment for a system of coplanar forces.

12. Deduce the condition for which a particle is in equilibrium at a point on a smooth plane curve.

13. Deduce the resultant of a system of concurrent coplanar forces.

14. Show that a system of coplanar forces can be reduced to a single force acting at a point in the plane together with a single couple.

15. Prove a necessary and sufficient conditions that a system of coplanar forces will be in equilibrium are
    (i) sum of resolved parts of the forces along any two mutually perpendicular direction will vanish.
    (ii) sum of the moments of all the forces about any point will vanish.

16. Derive the equation of line of action of resultant of a system of coplanar forces.

17. If the algebraic sum of moments of a system of coplanar forces about any three non-collinear points in the plane of the forces vanish separately, then show that the system is in equilibrium.

18. What do you mean by astatic equilibrium and astatic centre ?

19. A system of coplanar forces will be in equilibrium. Derive the condition for which this equilibrium will be astatic. Also fine the astatic centre.

20. If each force of a system of coplanar forces be rotated through an angle θ in its plane in the same sense, then show that their resultant is also rotated through the same angle.

21. A uniform beam of length $2a$, is rests against a smooth vertical plane over a smooth peg at a distance b from the plane. If θ be the inclination of the beam to the vertical, show that $a\sin^3\theta = b$.

22. A uniform rod of length $2a$, is rests against a smooth vertical wall and upon a smooth peg at a distance b from the wall. If θ be the inclination of the road to the horizontal, show that $a\cos^3\theta = b$.

23. A uniform rod of weight W rests with its ends in contact with two smooth planes inclined at an angles α and β respectively to the horizontal. If θ be the inclination of the road to the vertical, show that $2\cot\theta = \cot\beta \sim \cot\alpha$.

24. A uniform rectangular lamina rests vertically in equilibrium with its sides $2a,\ 2b\ \ (a > b)$ on two smooth pegs in the same horizontal line at distance c apart. Prove that the longer side makes with the horizontal an angle θ given by $c\cos 2\theta = a\cos\theta - b\sin\theta$.

25. The height of a cone is h and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg; show that, if the cone rests with its axis horizontal, the length of the string must be $\sqrt{h^2 + 4r^2}$.

26. A solid cone of height h and semi vertical angle α, is placed with its base against a smooth vertical wall and is suspended by a string attached to its vertex and to a point in the wall, show that the greatest possible height of the string is $h\sqrt{1 + \frac{16}{9}\tan^2\alpha}$.

27. Two equal smooth spheres, each of weight W and radius r are placed inside a hollow cylinder open at both ends which rests on a horizontal plane; if $a(< 2r)$ be the radius of the cylinder, show that the least weight it can have so as not to upset is $2W\left(1 - \frac{r}{a}\right)$.

28. A square frame $ABCD$ of four equal jointed rods hangs from A, the shape being maintained by a string joining mid-points of AB, BC. Prove that the ratio of the tension of the string to the reaction at C is $\frac{8}{\sqrt{5}}$.

29. The couple components of a system of coplanar forces when reduced with respect to two different bases O and O' are G and G' respectively. Show that the couple components when the system is reduced with respect to the middle point of OO' is $\frac{1}{2}(G + G')$.

30. The moments of the resultant R of a system of coplanar forces about three points $O,\ A$ and B lying in the plane of the forces are $G,\ G+J$ and $G+K$ respectively. Referred to O as origin, if the polar coordinates of A and B are (r,θ) and (ρ,ϕ) respectively, prove that $R^2\sin^2(\theta-\phi) = \frac{J^2}{r^2} + \frac{K^2}{\rho^2} - 2\frac{JK}{r\rho}\cos(\theta-\phi)$.

31. The moments of a system of forces about the points $(0,0), (a,0), (0,a)$ are $aW, 2aW, 3aW$ respectively. Find the components of their resultant parallel to the rectangular coordinate axes and the equation to its line of action.

32. A system of coplanar forces has the total moments $H, 2H$ respectively about the points whose coordinates are $(2a,0), (0,a)$ referred to the fixed rectangular axes. The total resolved part of the forces along the line $y = x$ vanishes. Find the points in which the line of action of the resultant meet the coordinate axes.

33. A system of coplanar forces has the total moments $H, 2H$ and $3H$ about the points $(0,0), (0,1)$ and $(2,4)$ respectively. Find the magnitude and the equation of the line of action of the resultant.

34. The straight line $4x + 3y = 5$ meet the rectangular axes OX and OY at the point A and B respectively. If the forces P, Q, R act along the lines OB, OA and AB, find the magnitude of the resultant and the equation of the line of action.

35. Three forces P, Q, R act along the sides of a triangle formed by the lines $x+y = 3,\ 2x+y = 1,\ x-y = -1$. Find the equation of the line of action of the resultant.

36. The forces P, Q, R act along the sides of the triangle formed by the lines $x + y = 1,\ y - x = 1,\ y = 2$. Find the equation of the line of action of the resultant.

37. A system of coplanar forces is equivalent to a couple of moment M, and if the forces are turned through a right angle, about their respective point of application in the same sense, they are equivalent to a couple N. Prove that when each force is turned about its point of application through an angle α in the same sense, the system will be in equilibrium if $M + N\tan\alpha = 0$.

38. Show that three coplanar forces P, Q, R acting at a point A, B, C are in astatic equilibrium if they meet at a point on the circum circle of the triangle ABC and if $P : Q : R = a : b : c$, where a, b, c are the sides of the triangle ABC.

An arbitrary force system in space:

1. Define (a) moment of a force about a line. (b) Couple and the axis of a couple.
2. State Varignon's theorem on moment for a system of forces.
3. Prove that if two forces act at a point, the algebraic sum of their moments about any line in space is equal to the moment of their resultant about the same line.
4. How can we determine the axis of any two couples acting on a body ?
5. Show that a system of forces acting on a rigid body can be reduced to a single force acting at an arbitrarily chosen point of the body together with a single couple.
6. If a system of forces acting on a rigid body be reduced to a single force R acting at an arbitrarily chosen point of the body and a single couple G, then show that Rb is invariant but not G.
7. Show that a force and a couple can not produce equilibrium.
8. Prove a necessary and sufficient conditions that a system of forces acting on a rigid body will be in equilibrium are (i) sum of resolved parts of the forces along any three mutually perpendicular lines should separately vanish. (ii) sum of the moments of all the forces about these lines should separately vanish.
9. Deduce the condition of equilibrium of a system of forces acting on a rigid body by using principle of virtual work.
10. (a) What is poinsot's central axis? (b) Define **Wrench, Pitch, Intensity** and **Screw.**
11. Show that any system of forces acting on a rigid body can be reduced to a single force and a couple whose axis lies along the line of action of the force.
12. Show that any system of forces acting on a rigid body can have only one central axis.
13. Derive the condition that a system of forces acting on a rigid body may have a single resultant.
14. If a system of forces acting on a rigid body is equivalent to six components $X, Y, Z : L, M, N$ of the system, then show that the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariants of the system.
15. Derive the equation of the central axis of a given system of forces acting on a rigid body.
16. OA, OB, OC are edges of a cube of side $2ft.$ and $OO',\ AA',\ BB', CC'$ are its diagonals; along $OB',\ O'A,\ BC$ and $C'A'$ act forces equal to 2 $lbs.wt.$, 6 $lbs.wt.$, 4 $lbs.wt.$ and 3 $lbs.wt.$ respectively. Prove that they are equivalent to a force $\sqrt{55}\ lbs.wt.$ at O in a direction whose direction cosines are proportional to $-1,\ -10,\ 3$ together with a couple $\sqrt{134}\ lbs.wt.$ whose axis has direction cosines proportional to $7,\ 3,\ -3$.
17. Six forces each equal to P, act along the edge of a cube, taken in order, which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}\ aP$, where a is the edge of the cube.
18. OA, OB, OC are edges of a cube of side a and $OO',\ AA',\ BB', CC'$ are its diagonals; along $OB',\ O'A,\ BC$ and $C'A'$ act forces equal to $P,\ 2P,\ 3P$ and $4P$ respectively. Prove that they are equivalent to a force $\sqrt{35}\ P$ at O in a direction whose direction cosines are proportional to $-3,\ -5,\ 6$ together with a couple $\frac{\sqrt{114}}{2}\ P$ whose axis has direction cosines proportional to $7,\ -2,\ 2$.
19. OA, OB, OC are edges of a rectangular parallelopiped of sides 6 *inches*, 15 *inches* and 8 *inches* respectively, and $OO',\ AA',\ BB', CC'$ are its diagonals. Calculate the wrench of the forces -130 from B' to O, 68 from A to O' and 68 from B to C.
20. Forces P, Q, R act along three non-intersecting edges of a cube; find the equation of the central axis.
21. Equal forces act along the axes and along the straight line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$; find the equation of the central axis.

22. Find the equation of the central axis for the system of forces P, Q, R acting along the straight lines $y = 0, z = 0;\ x = a, z = a;\ x = 0, y = a$.

23. If the system consist of two forces, one of which acts along z-axis and six components of the system are $X,\ Y,\ Z;\ L,\ M,\ N$; show that the force along z-axis is $\frac{LX+MY+NZ}{N}$.

24. A single force is equivalent to the component forces $X,\ Y,\ Z$ along the axes of coordinates and to a couples $L,\ M,\ N$ about these axes; show that the magnitude of the single force is $\sqrt{X^2 + Y^2 + Z^2}$ and the equation of the line of action is $\dfrac{yZ - zY}{L} = \dfrac{zX - xZ}{M} = \dfrac{xY - yX}{N} = 1$.

25. A system of forces is given by its six components $X,\ Y,\ Z;\ L,\ M,\ N$. If the system be replaced by two forces one acting along x-axis and another force, then show that the magnitude of the forces are $\frac{LX+MY+NZ}{L}$ and $\dfrac{\sqrt{L^2(Y^2 + Z^2) + (MY + NZ)^2}}{L}$. if

26. Three forces each equal to P act on a body; one at the point $(1, 0, 0)$ parallel to oy, the second at the point $(0, 1, 0)$ parallel to oz, and the third at the point $(0, 0, 1)$ parallel to ox; the axes being rectangular, find the equation of the central axis.

27. Three forces each equal to P act on a body; one at the point $(a, 0, 0)$ parallel to oy, the second at the point $(0, b, 0)$ parallel to oz, and the third at the point $(0, 0, c)$ parallel to ox; the axes being rectangular, find the resultant wrench in magnitude and position.

28. OA, OB, OC are three coterminous edges of a cube and $OO',\ AA',\ BB', CC'$ are its diagonals; along $BC',\ CA',\ AB'$ and OO' act forces equal to $X,\ Y,\ Z$ and R respectively. Show that they are equivalent to a single resultant if $R(X + Y + Z) + \sqrt{3}\ (XY + YZ + ZX)$.

29. Forces $P,\ Q,\ R,\ P,\ Q,\ R$ act along the edges $BC,\ CA,\ AB,\ AD,\ BD,\ CD$ of a regular tetrahedron $ABCD$. Show that they are equivalent to a wrench of pitch $\frac{a}{2\sqrt{2}}$, where a is the length of an edge of the tetrahedron.

30. Prove that if equal forces act along the edges $BC,\ CA,\ AB,\ DA,\ DB,\ DC$ of a regular tetrahedron $ABCD$, the central axis is perpendicular from D to the plane ABC and the pitch of the equivalent wrench is $\frac{a}{2\sqrt{2}}$, where a is the length of an edge of the tetrahedron.

31. Forces $X,\ Y,\ Z$ act along the three lines $y = b, z = -c;\quad z = c, x = -a;\quad x = a, y = -b$ respectively; show that they will have a single resultant if $\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0$, and equation of the line of action of the resultant is given by any two of the three $\dfrac{y}{Y} - \dfrac{z}{Z} - \dfrac{a}{X} = 0,\ \ \dfrac{z}{Z} - \dfrac{x}{X} - \dfrac{b}{Y} = 0,\ \ \dfrac{x}{X} - \dfrac{y}{Y} - \dfrac{c}{Z} = 0$.

32. Forces $X,\ Y,\ Z$ act along the three lines given by the equations $y = 0,\ z = c;\ z = 0,\ x = a; x = 0,\ y = b$; prove that the pitch of the equivalent wrench is $\dfrac{aYZ + bZX + cXY}{X^2 + Y^2 + Z^2}$. If the wrench reduced to a single force, show that the line of action of the force must lie on the hyperboloid $(x - a)(y - b)(z - c) = xyz$.

33. Three forces act along the straight lines $x = 0,\ y - z = a;\ y = 0,\ z - x = a; z = 0,\ x - y = a$. Show that they can not reduce to a couple. Prove also that if the system reduces to a single force, its line of action must lie on the surface $x^2 + y^2 + z^2 - a^2 = 2(yz + zx + xy)$.

34. Two equal forces act along each of the straight lines $\dfrac{x \pm a\cos\theta}{a\sin\theta} = \dfrac{y - b\sin\theta}{\pm b\cos\theta} = \dfrac{z}{c}$; show that their central axis, for all values of θ, must lie on the surface $y\left(\dfrac{x}{z} + \dfrac{z}{x}\right) = b\left(\dfrac{a}{c} + \dfrac{c}{a}\right)$.

35. Forces acting along the generators of the same system of a hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, are in equilibrium if they would be in equilibrium when acting at a point in the same directions.

36. Two equal forces act along the generators of the same system of the hyperboloid $\dfrac{x^2 + y^2}{a^2} - \dfrac{z^2}{b^2} = 1$, and cut the plane $z = 0$ at the extremities of perpendicular diameter of the circle $x^2 + y^2 = a^2$; Show that the pitch of the equivalent wrench is $\dfrac{a^2 b}{a^2 + 2b^2}$.

37. Forces $P,\ Q,\ R$ act along the three mutually perpendicular generators of the same system of the hyperboloid $\dfrac{x^2+y^2}{2a^2}-\dfrac{z^2}{a^2}=1$, the positive direction of the forces being towards the same side of the xy-plane. Prove that the pitch of the equivalent wrench is $\dfrac{2a(PQ+QR+RP)}{P^2+Q2+R^2}$.

38. Two forces $P,\ Q$ act along the straight lines whose equations are $y=x\tan\alpha,\ z=c$ and $y=-x\tan\alpha,\ z=-c$ respectively. Show taht their central axis is $y=x\left(\dfrac{P-Q}{P+Q}\right),\ \dfrac{z}{c}=\dfrac{P^2-Q^2}{P^2+Q^2+2PQ\cos 2\alpha}$. For all values of P and Q, prove that this line is a generator of the surface $(x^2+y^2)z\sin 2\alpha=2cxy$.

39. A force P acts along the axis of x and another force nP acts along the generator of the cylinder $x^2+y^2=a^2$; show that the central axis lies on the cylinder $n^2(nx-z)+(n^2+1)^2y^2=n^4a^2$.

40. A force F acts along the axis of z and another force mF acts along the straight line, intersecting the axis of x at a distance from the origin and parallel to the plane of yz. Show that as this line turns round the axis of x, the central axis of the force system generates the surface $\{m^2z^2+(m^2-1)y^2\}(c-x)^2=x^2z^2$.

41. Two forces act, one along the straight line $y=0, z=0$ and the other along the straight line $x=0, z=c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is $(x^2+y^2)z=cy^2$.

42. A force parallel to the axis of z acts at the point $(a,0,0)$ and an equal force perpendicular to the axis of z act at the point $(-a,0,0)$. Show that the central axis of the system lies on the surface $z^2(x^2+y^2)=(x^2+y^2-ax)^2$.

43. Two forces $2P,\ P$ act along the straight lines whose equations are $y=x\tan\alpha,\ z=c$ and $y=-x\tan\alpha,\ z=-c$ respectively. Find the equation of the central axis.

44. Equal forces act along the coordinate axes and along the straight line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$; find the equation of the central axis of the system.

45. If P and Q be two non-intersecting forces whose directions are perpendicular, show that the ratio of the distance of the central axis from their lines of action is Q^2 to P^2.

46. Show that the wrench $(X,\ Y,\ Z;\ \ L,\ M,\ N)$ is equivalent to two forces, one along the line $x=y=z$ and other along the straight line given by $x(Y-Z)+y(Z-X)+z(X-Y)=L+M+N$.

47. If a force $(X,\ Y,\ Z)$ acts along a generator of the hyperbolic paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\dfrac{2z}{c}$ and be equivalent to an equal force $(X,\ Y,\ Z)$ at the origin together with a couple $(L,\ M,\ N)$, show that $aL\pm bM=0,\ bX\pm aY=0$ and $cN\pm abz=0$.

48. $OBDC$ is a rectangle such that $CD=b, OC=c$; also OA is perpendicular to its plane; along $OA,\ CD,\ BD$ act forces $X,\ Y,\ Z$. Show that the component force R and couple of the resultant wrench are $\sqrt{X^2+Y^2+Z^2}$ and $\dfrac{X(bZ-cY)}{\sqrt{X^2+Y^2+Z^2}}$. Show also that the equation of the central axis with $OA,\ OB,\ OC$ as axis is $\dfrac{x}{X}=\dfrac{y}{Y}-\dfrac{kZ}{XYR}=\dfrac{z}{Z}+\dfrac{kY}{XZR}$.

49. Show that the minimum distance between two forces which are equivalent to a given system (R,G) and which are inclined at angle 2θ is $\dfrac{2G}{R}\cot\theta$ and that the forces are then each equal to $\dfrac{1}{2}R\sec\theta$.

50. The axes of two wrenches are at right angles and the shortest distance between them is $2a$. Prove that the axis of the resultant wrench divides the shortest distance in the ratio $Q\{2aQ+(p-q)P\}:P\{2aP-(p-q)Q\}$, where $P,\ Q$ are the intensities of the wrenches, and $p,\ q$ are the pitches.

Equilibrium in the presence of sliding Friction force:

1. What is contact force between two bodies? Name different types of contact forces with example.

2. (a) State Coulomb's laws of static friction? (a) State Coulomb's laws of dynamic friction?

3. Define: (a) Friction. (b) Limiting Friction. (c) Coefficient of friction.
   (d) Angle of Friction and Cone of Friction.

4. State the laws of (a) Static Friction. (b) Limiting Friction. (c) Dynamical Friction.

5. What is equilibrium and equilibrium region?

6. A particle is constrained to rest on a rough plane curve under the action of forces $X,\ Y$ parallel to the axes. Show that in all the position of equilibrium $\left(X+Y\dfrac{dy}{dx}\right)^2 \le \mu^2\left(X\dfrac{dy}{dx}-Y\right)^2$, μ being the coefficient of friction.

7. A particle is constrained to rest on a rough plane curve $f(x,y)=0$ under the action of forces $X,\ Y$ parallel to the axes. Show that in all the position of equilibrium $(Xf_x+Yf_y)^2 \ge \cos^2\lambda\left(X^2+Y^2\right)\left(f_x^2+f_y^2\right)$, λ being angle of friction.

8. A particle is constrained to rest on a rough space curve under the action of forces $X,\ Y,\ Z$ parallel to the axes. Show that in all the position of equilibrium $\left(X\dfrac{dx}{ds}+Y\dfrac{dy}{ds}+Z\dfrac{dz}{ds}\right)^2 \le \dfrac{\mu^2}{1+\mu^2}\left(X^2+Y^2+Z^2\right)$, μ being coefficient of friction.

9. A particle is constrained to rest on a rough surface under the action of given forces. Find the position of equilibrium.

10. If λ be the angle of friction, then show that the least force required to drag a heavy body of weight W along a rough horizontal plane is $W\sin\lambda$.

11. A heavy body is in limiting equilibrium on a rough inclined plane under the action of gravity only, show that the inclination of the plane is equal to the angle of friction.

12. A heavy particle is at rest on a rough curve in a vertical plane. If θ be the inclination of the tangent at a point to the vertical, show that $\cos^2\theta \le \dfrac{\mu^2}{1+\mu^2}$, μ being coefficient of friction.

13. Show that a particle can not rest at any point on the curve $x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$ which is higher than $2a\sin^2\lambda$ above the lowest point on the curve, where λ angle of friction and a is the radius of the generating circle of this curve.

14. A rough wire in the shape of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ (\ a>b\ )$ is placed with its major axis vertical. If μ be the coefficient of friction, then show that the depth below the highest point of position of limiting equilibrium of a bead which rest on the wire is $\left(a-\dfrac{a^2}{\sqrt{a^2+\mu^2b^2}}\right)$.

15. A heavy particle rests on a rough parabola $x^2=4ay$, y-axis is vertical. Show that the greatest altitude above the vertex of position of limiting equilibrium of the particle which rest on the parabola is μa^2.

16. A rough wire in the shape of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ (\ a>b\ )$. A small bead on the wire in the position of limiting equilibrium when at a distance $\dfrac{b}{2}$ from the major axis under the action of a force whose line of action passes through the centre of the ellipse. Show that the coefficient of friction is $\dfrac{\sqrt{3}\ (a^2-b^2)}{4ab}$.

17. The paraboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ is placed with its axis vertical and vertex uppermost; if μ be the coefficient of friction, show that a particle will rest on it at any point above its curve of intersection with the cylinder $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\mu$.

18. A particle rests on the surface $xyz = c^2$ under the action of a constant force parallel to the z-axis; show that the curve of intersection of the surface with the cone $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{\mu^2}{z^2}$ will seperate the part of the surface on which the equilibrium is possible from that on which it is not possible.

19. A rough surface is formed by the revolution of a rectangular hyperbola about a vertical asymptote; show that a particle will rest on it anywhere beyond its intersection with a certain circular cylinder.

20. A uniform ladder of weight W, inclined to the horizon at $45°$, rests with its upper extremity against a rough vertical wall and its lower extremity on the rough horizontal ground. Prove that the least horizontal force which will move the lower end towards the wall is just grater than $\dfrac{W(1+2\mu-\mu\mu')}{2(1-\mu')}$, where μ, μ' are coefficient of friction at the lower end and upper end respectively.

21. A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and the wall be both rough, the coefficients of friction being μ, μ' respectively, and if the ladder be at the point of slipping at both ends, show that the inclination θ of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$.

22. A uniform ladder of weight W rests on the rough horizontal ground and a smooth vertical wall inclined at an angle to the horizon. Prove that a man of weight w can climb to the top of the ladder without the ladder slipping if $\frac{W}{w} > \frac{2(1-\mu\tan\theta)}{2\mu\tan\theta-1}$.

23. A solid homogeneous hemisphere rests on a rough horizontal plane against a rough vertical wall. The coefficient of friction at the horizontal plane and vertical wall are μ, μ' respectively. Show that the least angle that the base of the hemisphere can make with the horizontal is $\sin^{-1}\left\{\frac{8}{3}\frac{\mu(1+\mu')}{1+\mu\mu'}\right\}$.

24. A solid homogeneous hemisphere rests on a rough horizontal plane against a smooth vertical wall. Show that if the coefficient of friction μ is less than $\frac{8}{3}$, then the angle θ between the base of the hemisphere and the horizontal must be lass than or equal to is $\sin^{-1}\left(\frac{8\mu}{3}\right)$.

25. A semi circular disc rest in a vertical plane with its curve edge on a rough horizontal plane, the coefficient of friction being μ. Show that the greatest angle that the bounding diameter ca make with the horizontal plane is $\sin^{-1}\left(\dfrac{3\pi}{4}\dfrac{\mu+\mu^2}{1+\mu^2}\right)$.

26. A solid hemisphere of weight W rests in limiting equilibrium with its curved surface on a rough inclined plane and its plane face is kept horizontal by a weight W' attached to a point in the rim. Prove that the coefficient of friction is $\dfrac{W'}{\sqrt{W(W+2W')}}$.

27. A uniform heavy elliptic wire, whose semi-axes a and b, is hung over a small rough peg. Show that, if the wire can be in equilibrium with any point of it in contact with the peg, the coefficient of friction must not be less than $\dfrac{a^2-b^2}{2ab}$.

28. A perfectly rough plane is inclined at angle α to the horizon, show that the least eccentricity of the ellipse which can rest on the plane, is $\sqrt{\dfrac{2\sin\alpha}{1+\sin\alpha}}$.

29. A sphere, whose radius is a and whose centre of gravity is at distance c from the centre, rests in limiting equilibrium on a rough plane inclined at an angle α to the horizon; show that it may be turned through an angle $2\cos^{-1}\left(\dfrac{a\sin\alpha}{c}\right)$ and still be in limiting equilibrium.

30. A heavy uniform circular disc of weight W and radius a lies on a rough horizontal plane of coefficient of friction μ. The end A of the diameter AB is held fixed while a pull P is exerted along a string attached to the end B. Show that the least value of P required to turn the disc about A is $\dfrac{16}{9}\mu W a^2$.

Unit-2:
Virtual Work:

1. Define workless constraint. Give example of workless constraint.
2. What do you mean by virtual displacement? How does it differ from real displacement?
3. Discuss the two types of virtual displacement:
   (i) Virtual displacement satisfying constraints. (ii) Virtual displacement violating constraints.
4. What is applied force. Is reaction of constraint an applied force?
5. Define virtual work.
6. If any number of forces act on a particle and the particle undergoes a small displacement, then show that the total work done by all the forces is equal to work done by their resultant.
7. State principle of virtual work: (a) for a single particle. (b) for a free rigid body. (c) for general rigid body.
8. State and prove the principle of virtual work for any system of coplanar forces acting on a rigid body.
9. State the converse of the principle of virtual work.
10. Name the forces which do not appear in equation of virtual work with justification.
11. Name the forces which do appear in equation of virtual work with justification.
12. What are the applications of the principle of virtual work?
13. Two equal uniform rods AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that, if 2θ be the angle between them, then $b\sin^3\theta = a\cos\theta$.
14. A rectangular board whose sides are a, b is supported with its plane vertical on two smooth pegs in the same horizontal line at distance c; prove that the angle θ made by the side a with the vertical when in equilibrium is given by the equation $2c\cos 2\theta = b\cos\theta - a\sin\theta$.
15. Two uniform rods, each of weight W and length a, are freely jointed at A, and each passes over a smooth peg at the same level. From A a weight W' is suspended. Show that in the position of equilibrium, the inclination θ of the rod to the horizon is given by $2aW\cos^3\theta = c(2W + W')$, c being the distance between the pegs.
16. Two uniform rods AB and BC of weights W and W' respectively are smoothly jointed at B and their middle points are joined across by a string. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the string is $\dfrac{(W + W')\cos A\cos C}{\sin B}$.
17. A smooth circular cylinder of radius b is fixed parallel to a smooth vertical wall with its axis at a distance c from the wall. A smooth uniform heavy rod of length $2a$ rests on the cylinder with one end on the wall and in a plane perpendicular to the wall; show that its inclination θ to the horizontal is given by $a\cos^2\theta + b\sin\theta = c$.
18. Four equal heavy uniform rods each of weight W are freely jointed so as to form a rhombus $ABCD$ which is freely suspended by the end A and a weight W' is attach to C. A light rod of negligible weight joins the middle points of AB, AD, keeping these inclined at an angle θ to AC. Show that the thrust in the light rod is $(4W + 2W')\tan\theta$.
19. A square framework, formed of heavy uniform rods of equal weights W jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Prove that the tension on this light rod is $4W$.

20. A rhombus $ABCD$ is formed of four equal uniform rods freely joined together and suspended from the point A; it is kept in position by a light rod joining the mid-points of BC and CD. If T be the thrust in this rod and W the weight of the rhombus, prove that $T = W \tan \frac{A}{2}$.

21. A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is $\dfrac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$.

22. Two uniform similar rods of same material PQ and QT of lengths $2l$ and $2l'$ respectively are rigidly united at Q and suspended freely from P. If they rest inclined at an angle α and β respectively to the vertical, prove that $(l^2 + 2ll') \sin \alpha = l'^2 \sin \beta$.

23. A heavy rod AB, of length $2l$, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions show that the polar equation of the curve, with C as origin, is $r = l + a\ cosec\ \theta$.

24. A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any position, then show that the equation of the curve is $x^{\frac{2}{3}} + (y-3)^{\frac{2}{3}} = a^{\frac{2}{3}}$.

25. One end of a beam rests against a smooth vertical wall and other end on a smooth curve in a vertical plane perpendicular to the wall. If the beam rests in all positions, show that the curve is an ellipse $\dfrac{x^2}{4a^2} + \dfrac{y^2}{a^2} = 1$ with major axis along the horizontal line described by the C.G. of the beam.

26. A heavy elastic string whose natural length is $2\pi a$ is placed around a smooth cone whose axis vertical and whose semi-vertical angle is α. If W be the weight and λ be the modulus of elasticity of the string; prove that it will be in equilibrium when in the form of a circle whose radius is $a\left(1 + \dfrac{W}{2\pi\lambda} \cot \alpha\right)$.

27. A smooth paraboloid of revolution is fixed with its axis vertical and vertex upwards; on it is placed a heavy elastic string of unstretched length $2\pi c$. When the string is in equilibrium; show that it rests in the form of a circle of radius $\dfrac{4\pi ac\lambda}{4\pi a\lambda - W}$, where W is the weight of the string, λ its modulus of elasticity and $4a$ the latus rectum of the generating parabola.

28. A rod AB is movable about a join at A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is $\dfrac{W \cos \alpha \cos \beta}{2 \sin(\alpha + \beta)}$, where W is the weight of the rod and α, β the inclination of the rod and the string to the horizontal.

29. A solid hemisphere is supported by a string fixed to a point to its rim and to a point on the smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan \phi = \dfrac{3}{8} + \tan \theta$.

30. A lamina in the form of an isosceles triangle rests with its plane vertical and its two equal sides each in contact with a smooth peg, the pegs being in the same horizontal plane. Prove that the axis of the triangle makes with the vertical the angle zero or $\cos^{-1}\left(\dfrac{h \sin \alpha}{3c}\right)$, h being the length of the axis, α the vertical angle and c is the distance between the pegs.

31. Five equal uniform rods, each of weight W are freely joined together to form a pentagon $ABCDE$, which is suspended from the join A and maintained in the shape of a regular pentagon by two strings joining A to C and D. Show that the tension of either string is $2W \cos 18^\circ$.

32. A regular pentagon $ABCDE$ is formed by five uniform heavy rods each of weight W freely jointed at their extremities. It is freely suspended from A and is maintained in its regular pentagonal form by a lighter rod joining B and E. Prove that the strees in this rod is $2W \cos 18^\circ$.

33. A quadrilateral $ABCD$ form of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the join A. A string joins A to C and is such that $\angle ABC = \frac{\pi}{2}$. Applying the principle of virtual work, show that the tension in the string is $(w+w')\sin^2\theta+w'$, where w is the weight of an upper rod and w' of a lower rod and $2\theta = \angle BAD$.

34. A square frame $ABCD$ of four equal jointed rods hangs from A, the shape being maintained by a string joining mid points of AB, BC. Using principle of virtual work prove that the ratio of the tension of the string to the reaction at C is $\dfrac{8}{\sqrt{5}}$.

35. A parallelogram $ABCD$ formed of four uniform rods freely jointed at the corners rests in equilibrium in a vertical plane with AB fixed horizontally. A is attached to the opposite corner C by a light string of length l. AC is the shorter diagonal and α is the acute angle of the parallelogram. Show that the tension in the string is $\dfrac{Wl\cot\alpha}{2a}$, where W is the weight of the rod and $AB = a$.

36. A heavy uniform rod AB, of length $2a$, rests with its ends in contact with two smooth inclined planes of inclinations α, β to the horizon. If θ be the inclination of the rod to the horizon, then using the principle of virtual work prove that $\tan\theta = \dfrac{1}{2}(\tan\alpha \sim \tan\beta)$.

37. Five weightless rods form a regular pentagon $ABCDE$, and the framework is stiffened by another weightless rods BD, CE. The system is placed in a vertical plane with CD on a horizontal table, and a weight W is hung from A. Prove that the thrust in BD or CE is $W\cot\dfrac{\pi}{5}$.

38. The middle points of opposite sides of a quadrilateral formed by four freely jointed weightless bars are connected by two light rods of length l and l' in a state of tension. If T, T' are the tensions in these rods, prove that $\dfrac{T}{l} + \dfrac{T'}{l'} = 0$.

39. $ABCD$ is a rhombus of freely jointed rods lying flat on a smooth table and P, Q are the middle points of AB, AD. Prove that if the system held in equilibrium by light strings joining P to Q and A to C, the tensions in these strings are in the ratio of 2 BD to AC.

40. Six equal bars smoothly jointed form a regular hexagon $ABCDEF$ which kept in shape by vertical strings joining the middle points of BC, CD and AF, FE respectively, the side AB being held in horizontal and CD uppermost. Prove that the tension of each string is three times the weight of a bar.

41. Two equal rods each of weight wl and length l, are hinged together and placed symmetrically over a smooth horizontal cylindrical peg of radius r. The lower ends are tied together by a string and the rods are left at the same inclination θ to the horizontal. Find the tension in the string. If the string is slack, show that θ satisfies the equation $2r(\tan^3\theta + \tan\theta) = l$.

42. Four uniform rods are jointed to form a rectangle $ABCD$. AB is fixed in a vertical position with A uppermost, and the rectangle is kept in shape by a string joining A to C. Show that the tension of the string is $\dfrac{W}{2}\dfrac{AC}{AB}$.

43. A frame consists of five weightless bars forming the sides of a rhombus $ABCD$ with the diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in AC is $\dfrac{P\cos 2\theta}{\sin\theta}$, where $\theta = \angle BAC$.

44. Two uniform straight rods PQ, $P'Q$, in all respects alike, are smoothly jointed at Q and P, P' carry small rings which slides on a smooth fixed parabolic wire whose axis vertical and vertex upwards. Prove that in symmetrical position of equilibrium the angle either rod makes with the horizontal is $\sin^{-1}\left\{\dfrac{aW}{l(W+w)}\right\}$, where W is the weight of the either rod, w of either ring, l length of the either rod and $4a$ latus rectum of the parabola.

45. Four uniform rods AB, BC, CD, DE each of length a and weight W are smoothly jointed together at their ends B, C and D and the ends A, E are smoothly jointed to fixed points at distance $2a$ apart in the same horizontal line. If AB, BC make angles θ and ϕ respectively with the horizontal when the system hangs in equilibrium, show by principle of virtual work that $3\cot\theta = \cot\phi$.

Stability of Equilibrium:

1. What do you mean by the term **equilibrium**? What are the different types of equilibrium?

2. Define: (*a*) Stable Equilibrium. (*b*) Unstable Equilibrium. (*c*) Neutral. (*d*) On the whole Unstable.

3. What do you mean by the term **Rocking Stones**?

4. What is conservative field of force? Sate a necessary and sufficient condition for a field of force to be conservative.

5. Prove that in a conservative field of force, the sum of Kinetic Energy and Potential Energy is constant for all position.

6. State and prove energy test of stability.

7. What is degrees of freedom? Prove energy test of stability of the system having one degrees of freedom.

8. Discuss the stability of the system when the gravity is the only external force.

9. A perfectly rough (sufficient friction to prevent sliding) body rests in equilibrium on another fixed body. The portion of two bodies in contact are spherical and of radii r_1 and r_2 respectively and also the line joining their centres in position of equilibrium is vertical. Show that the equilibrium is stable or unstable according as $\frac{1}{h} \gtrless \frac{1}{r_1} + \frac{1}{r_2}$, where h is the height of C.G of the upper body in the position of equilibrium above the point of contact. Also discuss the case when $r_1 r_2 = h(r_1 + r_2)$.

10. A perfectly rough heavy body rests in equilibrium on another fixed body. The parts of the bodies in contact are spherical and of radii r_1 and r_2 respectively and the surface of lower body concave upwards, the upper body is in equilibrium at the lowest. Supposing the line joining their centers in position of equilibrium is vertical, Show that the equilibrium is stable or unstable according as $\frac{1}{h} \lesseqgtr \frac{1}{r_1} - \frac{1}{r_2}$, where h is the height of C.G of the upper body in the position of equilibrium. Also discuss the case when $r_1 r_2 = h(r_2 - r_1)$.

11. A perfectly rough heavy body rests in equilibrium on a fixed body at the point A. The common normal at the point of contact A makes an angle α with the vertical. The center of gravity G of the upper body must lie on the vertical at A and h is the height of G above A. Show that the equilibrium is stable or unstable according as $\frac{\cos\alpha}{h} \lessgtr \frac{1}{\rho_1} + \frac{1}{\rho_2}$, where ρ_1, ρ_2 are the radii of curvature at A of the curve of intersection in which the two bodies are cut by the vertical plane of symmetry in which the displacement is given.

12. A uniform cubical box of edge a is placed on the top of a fixed sphere. Show that the least radius of the sphere for which the equilibrium will be stable is $\frac{a}{2}$.

13. A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

14. A hemisphere rests in equilibrium on a sphere of equal radius. Show that the equilibrium is unstable or stable according as the curved surface or plane surface of the hemisphere rests on the sphere.

15. A solid body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, cannot exceed $\sqrt{3}$ times the radius of the hemisphere.

16. A solid frustum of paraboloid of revolution, of height h and latus rectum $4a$, rests with its vertex of a paraboloid of revolution whose latus rectum is $4b$. Show that the equilibrium is stable if $h < \frac{3ab}{a+b}$.

17. A lamina in the form of a cycloid, whose generating circle is of radius a, rests on the top of another cycloid whose generating circle is of radius b, their vertices being in contact and their axes vertical. If h be the height of the centre of gravity of the upper cycloid above its vertex, show that the equilibrium is stable only if $h < \frac{4ab}{a+b}$ and is unstable if $h \geq \frac{4ab}{a+b}$.

18. A sphere of weight W and radius a, lies within a fixed spherical shell of radius b and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if $\dfrac{W}{w} > \dfrac{b-2a}{a}$ and if $\frac{W}{w}$ be equal to this ratio, the equilibrium is really stable.

19. A thin hemispherical bowl, of radius b and weight W, rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w. Show that the equilibrium is not stable unless $w \not< \dfrac{(a-b)W}{2b}$.

20. A lamina in the form of an isosceles triangle whose vertical angle is α, is placed on a sphere of radius r so that its plane is vertical and one of equal side is in contact with the sphere. Show that if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < \dfrac{3r}{a}$, where a is one of the equal sides of the triangle.

21. A solid ellipsoid, whose axes are of lengths $2a$, $2b$, $2c$ rests with its C-axis vertical on a rough horizontal plane. The centre of gravity is on the vertical axis at a distance h from the bottom vertex. Show that the equilibrium is stable if h is less than both $\frac{a^2}{c}$ and $\frac{b^2}{c}$.

22. A heavy uniform hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $\dfrac{R}{r} > (\sqrt{5}-1)$, the equilibrium is stable, whatever be the weight of the particle.

23. A uniform square board of mass M is supported in a vertical plane on two smooth pegs at the same horizontal level. The distance between the pegs ia a and the diagonal of the square is $D < 4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable if $4am > M(D-4a)$.

24. A heavy rod rests with its ends on two smooth inclined planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β and the centre of gravity of the rod divides it in the ratio $a:b$, find the position of equilibrium of the rod and show that it is unstable.

25. A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of a square. Find the position of equilibrium and show that it is stable.

26. A uniform square lamina rests in equilibrium in a vertical plane with two of its sides in contact with smooth pegs in the same horizontal line at distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by $c(\sin\theta + \cos\theta) = a$, $2a$ being the length of the side of the square. Discuss the stability of possible position of equilibrium.

27. A uniform rod of length $2l$, is attached by smooth rings at both ends to a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium is given by $\cos^2\theta = \dfrac{2a}{l}$; and that, if these positions exist, they are unstable.

28. A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation $\cos^4\dfrac{\theta}{2} = \dfrac{a}{2c}$, where $4a$ is the latus rectum and $2c$ length of the rod. Investigate also the stability of equilibrium in this position.

29. Two equal uniform rods of length l are firmly jointed at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in stable or unstable equilibrium according as $l\sin\alpha \gtrless 4r$.

30. A solid homogeneous sphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1}\left(\dfrac{3}{8}\right)$, and the plane is rough enough to prevent sliding. Find the position of equilibrium and show that it is stable.