CC9: Unit-2: Multivariate Calculus-II
Double Integration:

Change the order of integration in the following double integrals:

1. $\int_0^4 dx \int_{3x^2}^{12x} f(x,y)dy$
2. $\int_0^1 dx \int_{2x}^{3x} f(x,y)dy$
3. $\int_0^a dx \int_{\frac{a^2-x^2}{2a}}^{\sqrt{a^2-x^2}} f(x,y)dy$
4. $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx$
5. $\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{4ax-x^2}} f(x,y)dy$
6. $\int_0^2 dx \int_{2x}^{6-x} f(x,y)dy$
7. $\int_{\frac{a}{2}}^a dx \int_0^{\sqrt{2ax-x^2}} f(x,y)dy$
8. $\int_0^1 dx \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x,y)dy$
9. $\int_0^{2a} dx \int_{\frac{x^2}{4a}}^{3a-x} f(x,y)dy$
10. $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)dx$
11. $\int_0^1 dx \int_0^x f(x,y)dy + \int_1^2 dx \int_0^{2-x} f(x,y)dy$
12. $\int_0^1 dx \int_0^{x^2} f(x,y)dy + \int_1^3 dx \int_0^{\frac{2-x}{2}} f(x,y)dy$
13. $\int_{\frac{1}{3}}^{\frac{2}{3}} dx \int_{x^2}^{\sqrt{x}} f(x,y)dy$
14. $\int_{-1}^1 dx \int_{-x}^{1+x} f(x,y)dy$
15. $\int_0^1 dy \int_{1-y}^{1+y} f(x,y)dx$
16. $\int_0^1 dy \int_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x,y)dx$
17. $\int_2^4 dx \int_{\frac{4}{x}}^{\frac{20-4x}{8-x}} (4-y)dx$
18. $\int_0^1 dy \int_{y^2}^{3-2y} (12-3x-4)dx$
19. By changing the order of integration prove that $\int_0^a dx \int_0^x \frac{f'(y)dy}{\sqrt{(a-x)(x-y)}} = \pi\{f(\pi)-f(0)\}$.
20. By changing the order of integration prove that $\int_0^1 dy \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}} = \frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)$.
21. By changing the order of integration prove that $\int_0^1 dy \int_x^{\frac{1}{x}} \frac{ydy}{(1+xy)^2(1+y^2)} = \frac{\pi-1}{4}$.
22. By changing the order of integration prove that $\int_0^1 dy \int_x^{\frac{1}{x}} \frac{y^2dy}{(x+y)^2\sqrt{1+y^2}} = \sqrt{2}-\frac{1}{2}$.
23. Let f be a bounded function of x, y over rectangular region $R[a,b;c,d]$. Considering a partition P of $R[a,b;c,d]$, define lower sum $L(P;f)$ and upper sum $U(P;f)$. when f is integrable over R?
24. Let a function f be defined on $R[1,2\ ;3,5]$ by $f(x,y)=x+2y$, for $(x,y)\in R[1,2\ ;3,5]$. Find the lower integral sum and upper integral sum. Does $\iint_R f(x,y)dxdy$ exists?
25. Let a function f be defined on $R[0,1\ ;0,1]$ by $f(x,y) = \begin{cases} \frac{1}{2} & \text{when } y \text{ is rational} \\ x & \text{when } y \text{ is irrational} \end{cases}$

    (i) Does $\iint_R f(x,y)dxdy$ exists?

    (ii) Examine whether the iterated integrals $\int_0^1 dy \int_0^1 f(x,y)dx$ and $\int_0^1 dx \int_0^1 f(x,y)dy$ exist.

26. State the necessary and sufficient condition for the integrability of a f be a bounded function of x, y over rectangular region $R[a, b; c, d]$.

27. Prove that if the double integral exists, the two repeated integrals can not exist without being equal.

28. Prove that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$.

Evaluate the following integrals:

29. $\int_0^{\frac{\pi}{2}} \int_0^{\pi} \cos(x+y)\ dxdy$

30. $\int_0^1 \int_0^{1-y^2} \left\{(x-1)^2 + y^2\right\}\ dxdy$

31. $\int_0^1 dx \int_0^x \sqrt{4x^2 - y^2}\ dy$

32. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\ dxdy$

33. $\int_0^a dx \int_0^b xy(x^2+y^2)dy$

34. $\int_0^2 dx \int_{2x}^{6-x} x^2 y\ dy$

35. $\int_{-1}^1 \int_{-1}^1 \frac{dxdy}{\sqrt{x^2+y^2}}$

36. $\int_{-1}^1 \int_{-1}^1 |x+y| dxdy$

37. $\int_0^1 dx \int_0^1 xy(x-y)dxdy$

38. $\int_0^2 dx \int_y^{\sqrt{y}} (1+x+y)dy$

39. $\int_0^1 dy \int_{y^2}^{3-2y} (12-3x-4)dx$

40. $\int_0^a dx \int_{x^2}^x xy(x+y)\ dy$

41. Evaluate $\iint_R [x+y]\ dxdy$, where R is the rectangle bounded by $x=0,\ x=1;\ y=0,\ y=2$.

42. Prove that $\iint_R \sqrt{|y-x^2|}\ dxdy = \frac{1}{6}(3\pi+8)$, where R is the rectangle bounded by $x=-1,\ x=1;$ $y=0,\ y=2$.

43. Evaluate $\iint_R (y-x)\ dxdy$, where R is the rectangle in xy- plane bounded by $y=x-3,\ y=x+1;$ $3y+x=5,\ 3y+x=7$.

44. Prove that $\iint_R x^3y^3\ dxdy = \frac{1}{48}(b^4-a^4)(q^4-p^4)$, where R is the region bounded by $y^2=ax,\ y^2=bx;$ $x^2=py,\ x^2=qy$, where $0<a<b$ and $0<p<q$.

45. Use the transformation $u = \frac{x^2+y^2}{x},\ v = \frac{x^2+y^2}{y}$ to evaluate the integral $\iint_R xy\ dxdy$, where R is the region common to the circles $x^2+y^2=x,\ x^2+y^2=y$.

46. Prove that $\iint_R \sqrt{xy(1-x-y)}\ dxdy = \frac{2\pi}{105}$, where R is the triangle bounded by the lines $x=0, y=0$ and $x+y=1$.

47. Evaluate $\iint_R \sqrt{2a^2-2a(x+y)-(x^2+y^2)}\ dxdy$, the region R of integration is the circle with center at (a,a) and radius $2a$.

48. Evaluate $\iint_R \sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}\ dxdy$, R is the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

49. Show that the integral $\iint_R e^{\frac{y-x}{y+x}}\, dxdy$, where R is the triangular region with vertices $(0,0),(0,1)$ and $(1,0)$ is $\frac{1}{4}(e-\frac{1}{e})$.

50. Evaluate $\int\int \frac{x^{\frac{1}{2}}y^{\frac{1}{3}}}{(1-x-y)^{\frac{2}{3}}}\, dxdy$ over the region bounded by the lines $x=0, y=0, x+y=1$.

51. Show that $\iint_R \frac{dxdy}{(1+x^2+y^2)^2}$, where R is the triangular region with vertices $(0,0),(2,0)$ and $(1,\sqrt{3})$ is $\frac{\sqrt{3}}{2}\tan^{-1}\frac{1}{2}$.

52. Prove that $\iint_R \sqrt{x^2+y^2}\, dxdy$, where R is the region in xy-plane bounded by the concentric circles $x^2+y^2=1$ and $x^2+y^2=4$ is $\frac{14}{3}\pi$.

53. Using the transformation $x=u(1+v),\ y=v(1+u)$, show that $\int_0^2\int_0^x \frac{dxdy}{\sqrt{(x+y+1)^2-4xy}}\, dxdy = \log\frac{4}{\sqrt{e}}$.

54. Show that $\int_0^1 dx \int_0^x \sqrt{x^2+y^2}\, dy = \frac{1}{6}\{\sqrt{6}+\log(1+\sqrt{2})\}$ by transforming it into polar coordinates.

55. Show that $\int_0^\pi\int_0^\pi |\cos(x+y)| dxdy = 2\pi$ by using the substitution $x=u-v, y=v$.

56. Evaluate $\iint_R \sin\left(\frac{x-y}{x+y}\right) dxdy$, where R is the region in xy-plane bounded by $x=0, y=0$ and $x+y=1$.

57. Prove that $\iint_R \sin x \sin y \sin(x+y) dxdy = \frac{\pi}{16}$, where R is the region in xy-plane bounded by $x=0, y=0$ and $x+y=\frac{\pi}{2}$.

58. Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi d\theta$, by using the substitution $x=\sin\phi\cos\theta,\ y=\sin\phi\sin\theta$.

59. Evaluate the integral $\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{4ax-x^2}} \left(1+\frac{y^2}{x^2}\right) dy$ by changing the coordinates to r,θ, where $x=r\cos^2\theta, y=r\sin\theta\cos\theta$.

60. Show that $\iint_R \frac{dxdy}{xy} = \log\frac{a'}{a}\log\frac{b'}{b}$, where R is the region bounded by four circles $x^2+y^2=ax, x^2+y^2=a'x,\ x^2+y^2=bx$ and $x^2+y^2=b'x$.

61. Show that $\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} f(1-\sin\theta\cos\phi)\sin\theta d\theta = \frac{\pi}{2}\int_0^1 f(x)\, dx$.

62. If $m \geq 0$, prove that $\iint_R \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^m f(px+qy)dxdy = \beta(\frac{1}{2}, m+1)ab\int_{-1}^1 (1-x^2)^{m+\frac{1}{2}} f(kx)dx$, where R is the region in xy-plane bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $k=\sqrt{p^2a^2+q^2b^2}$.

Surface area by using multiple Integration:

1. Show that the surface area of the part of the surface of the sphere $x^2+y^2+z^2=a^2$ which is cut out by the cylinder $x^2+z^2=ax$ is $2(\pi-2)a^2$.

2. Show that the surface area of the part of the surface of the sphere $x^2+y^2+z^2=4a^2$ enclosed by the cylinder $(x^2+y^2)^2=2a^2(2x^2+y^2)$ is $16(\pi-\sqrt{2})a^2$.

3. Find the area of the surface of the sphere $x^2+y^2+(z-2)^2=4$ which lies outside the paraboloid $x^2+y^2=3z$.

4. Show that the surface area of the part of the surface of the cone $z^2=x^2+y^2$ which is cut out by the cylinder $z^2=2py$ is $2\sqrt{2}\pi p^2$.

5. Show that the surface area of the part of the surface of the cylinder $x^2+y^2=a^2$ which is cut out by the cylinder $x^2+z^2=a^2$ is $8a^2$.

6. Show that the surface area of the part of the surface of the cone $z^2+y^2=x^2$ inside the cylinder the cylinder $x^2+y^2=a^2$ is $2\pi a^2$.

7. Show that the surface area of the part of the surface of the cone $z^2+y^2=x^2$ cut off by the cylinder the cylinder $x^2-y^2=a^2$ and the planes $y=b, y=-b$ is $8\sqrt{2}\ ab$.

8. Show that the surface area of the part of the surface $z=xy$ cut off by the cylinder the cylinder $x^2+y^2=a^2$ is $\frac{2\pi}{3}\{(1+a^2)^{\frac{3}{2}}-1\}$.

9. Show that the surface area of the part of the cone $z^2=x^2+y^2$ inside the cylinder the cylinder $x^2+y^2=2x$ is $2\sqrt{2}\ \pi$.

10. Show that the surface area of the part of the surface of the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ inside the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k$ is $\frac{2}{3}\pi\{(1+k)^{\frac{3}{2}}-1\}ab$.

11. Evaluate $\iint_S \left(z+2x+\frac{4}{5}y\right)\ dS$, where S is the portion of the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$, lying in the first octant.

12. Evaluate $\iint_S xyz\ dS$, where S is the portion of the plane $x+y+z=1$, lying in the first octant.

13. Evaluate $\iint_S x\ dS$, where S is the portion of the sphere $x^2+y^2+z^2=a^2$, lying in the first octant.

14. Evaluate $\iint_S \sqrt{a^2-x^2-y^2}\ dS$, where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$.

15. Evaluate $\iint_S x^2y^2\ dS$, where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$.

16. Evaluate $\iint_S x^2y^2z\ dS$, where S is the positive side of the lower half of the sphere $x^2+y^2+z^2=a^2$.

17. Evaluate $\iint_S z^2\ dS$, where S is the outer side of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Triple Integration:

Evaluate the following triple integral:

1. $\int_0^{3a}\int_0^{2a}\int_0^{a}(x+y+z)dxdydz$
2. $\int_0^{1}\int_{y^2}^{1}\int_0^{1-x} xdzdxdy$
3. $\int_0^{a}\int_{y^2}^{x}\int_0^{x+y} e^{x+y+z}dzdydx$
4. $\int_0^{2}\int_0^{z}\int_0^{\sqrt{3}x} \frac{x}{x^2+y^2}dzdydx$
5. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_a^{b}(y^2+z^2)dzdydx$
6. $\int_0^{1}dx\int_0^{1-x}dy\int_0^{1-x-y}(1+x+y+z)^4dz$
7. Show that $\iiint_E(1+x+y+z)^2\ dxdydz=\frac{31}{60}$, where E is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.
8. Show that $\iiint_E\frac{dxdydz}{(1+x+y+z)^3}=\frac{1}{16}\log\left(\frac{256}{e^5}\right)$, where E is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.
9. Show that $\iiint_E x^2y^2z^2(x+y+z)\ dxdydz=\frac{1}{50400}$, where E is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.
10. Evaluate $\iiint_E x^{\alpha}y^{\beta}z^{\gamma}(1-x-y-z)^{\lambda}\ dxdydz$; $\alpha,\beta,\gamma,\lambda>-1$, where E is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.
11. Show that $\iiint_E\frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}=\frac{\pi^2}{8}$, where $E=\{(x,y,z):x^2+y^2+z^2<1\}$.
12. Show that $\iiint_E(x^2+y^2+z^2)\ dxdydz=\frac{4\pi}{5}$, where E is the volume of the sphere $x^2+y^2+z^2\le 1$.
13. Evaluate $\iiint_E\sqrt{\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}}\ dxdydz$, where E is the positive octant of the sphere $x^2+y^2+z^2\le 1$.
14. Show that the mass of the solid in the form of the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point (x,y,z) being xyz.
15. Evaluate $\iiint_E e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\ dxdydz$, where E is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1,\quad a,b,c>0$.
16. Evaluate $\iiint_E\sqrt{a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2}\ dxdydz$, where $E=\left\{(x,y,z):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$
17. Show that $\iiint_E\frac{dxdydz}{x^2+y^2+(z-2)^2}=\pi\left(2-\frac{3}{2}\log 3\right)$, where $E=\{(x,y,z):x^2+y^2+z^2\le 1\}$.
18. Show that $\iiint_E\frac{dxdydz}{x^2+y^2+(z-\frac{1}{2})^2}=\pi\left(2+\frac{3}{2}\log 3\right)$, where $E=\{(x,y,z):x^2+y^2+z^2\le 1\}$.
19. Show that $\iiint_E(ax+by+cz)dxdydz=\frac{4}{15}\pi(a^2+b^2+c^2)$, where $E=\{(x,y,z):x^2+y^2+z^2\le 1\}$.
20. Show that $\iiint_E(lx^2+my^2+nz^2)^2dxdydz=\frac{4}{15}\pi(l+m+n)a^5$, where $E=\{(x,y,z):x^2+y^2+z^2\le a^2\}$.

Volume of a solid by using multiple Integration:

1. Prove that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ay$ is $\frac{2}{9}(3\pi-4)a^3$.

2. Prove that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ is $\frac{2}{9}(3\pi-4)a^3$.

3. Prove that the volume common to the surface $y^2+z^2=4ax$ and the cylinder $x^2+y^2=2ax$ is $\frac{2}{3}(3\pi+8)a^3$.

4. Prove that the volume common to the cylinders $x^2+y^2=a^2$ and the cylinder $x^2+z^2=a^2$ is $\frac{16}{3}a^3$.

5. Using surface integral show that the volume of the sphere $x^2+y^2+z^2=a^2$ is $\frac{4}{3}\pi a^3$.

6. Using surface integral show that the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi abc$.

7. Show that the volume of the solid bounded by the $x^2+y^2+z^2=4$ and the surface of the paraboloid $x^2+y^2=3z$ is $\frac{19}{6}\pi$.

8. Prove that the volume common to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the cylinder $x^2+y^2=ay$ is $\frac{2}{9}(3\pi-4)a^2b$.

9. Prove that the volume of the solid bounded by the parabolic cylinder $z=4-y^2$ and bounded below by the elliptic paraboloid $x^2+3y^2=z$ is 4π.

10. Compute the volume of the solid bounded by xy-plane, paraboloid $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{a}$.

11. Prove that the volume included between the cylinder $x^2+y^2=a^2$ and the elliptic paraboloid $\frac{x^2}{p}+\frac{y^2}{q}=2z$ and xy-plane is $\frac{p+q}{8pq}\pi a^4$.

12. Find the volume of the region bounded by the plane $z=x+y$ and the paraboloid $x^2+y^2=cz$.

13. Show that the volume of the region bounded by the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$ and three coordinate planes is $\frac{abc}{90}$.

14. Find the volume of the solid bounded above by the surface $2x^2+4y^2+z=4$ and bounded below by the surface $2x^2+4y^2-4z=4$.

15. Find the volume of the solid bounded by the surfaces $x^2+y^2=2az$, $x^2+y^2-z^2=a^2$ and $z=0$.

16. Compute the volume of the solid bounded by the surface $(x^2+y^2+z^2)^2=a^3x$.

17. Determine the volume of the solid bounded by the surfaces $z=x+y,\ xy=1,\ xy=2,\ y=x,\ y=2x,\ z=0$, where $x>0,\ y>0$.

18. Compute the volume of the solid bounded by the paraboloid $x^2+y^2=a(a-2z),\ z\geq 0$ and sphere $x^2+y^2+z^2=a^2$.

Differentiation under the sign of Integration:

1. Show that $\displaystyle\int_0^{\pi} \log(1+a\cos x)dx = \pi \log \frac{1}{2}(1+\sqrt{1-a^2}),\ |a|<1.$

2. Show that $\displaystyle\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}dx = \pi \sin^{-1} a,\ |a|<1.$

3. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x}dx = \pi(\sqrt{1+a}-1),\ a>-1.$

4. Show that $\displaystyle\int_0^{\pi} \frac{\log(1+\sin\alpha\cos x)}{\cos x}dx = \pi\alpha.$

5. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx = \frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right).$

6. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \log(1-x^2\sin^2\theta)d\theta = \pi\log\frac{1}{2}(1+\sqrt{1-x^2}) = \int_0^{\frac{\pi}{2}} \log(1-x^2\cos^2\theta)d\theta, \quad \text{for } |x|<1.$

7. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \log(1-e^2\sin^2\theta)d\theta = \pi\log\frac{1}{2}(1+\sqrt{1-e^2}), \quad \text{for } 0<e^2<1.$

   Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \log(\sin\theta)d\theta.$

8. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta+b^2\sin^2\theta)d\theta = \pi\log\frac{1}{2}(a+b), \text{ for } a>0,\quad b>0.$

9. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi\sin^{-1}\left(\frac{b}{a}\right), \quad \text{for } b<a.$

10. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{|ab|dx}{(a^2\cos^2 x+b^2\sin^2 x)} = \frac{\pi}{2}, \quad a,b\in\mathbb{R}-\{0\},$

    Hence show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{|ab|dx}{(a^2\cos^2 x+b^2\sin^2 x)^2} = \frac{\pi(a^2+b^2)}{4|ab|^3}.$

11. Show that $\displaystyle\int_0^{\pi} \log(1-2a\cos x+a^2)dx = \pi\log(a^2), \text{ where } |a|>1.$

12. Show that $\displaystyle\int_0^{\theta} \log(1+\tan\theta\tan x)dx = \theta\log(\sec\theta), \text{ where } -\tfrac{\pi}{2}<\theta<\tfrac{\pi}{2}.$

13. Show that $\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cos^{-1}(\cos\alpha\ cosec\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha), \quad 0<\alpha<\frac{\pi}{2}.$

14. Show that $\displaystyle\int_0^{a} \frac{\log(1+ax)}{1+x^2}dx = \frac{1}{2}\log(1+a^2)\tan^{-1} a, \quad a>0.$

15. Show that $\displaystyle\int_0^{1} \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}dx = \frac{\pi}{2}\log(a+\sqrt{1+a^2}).$

16. Show that $\displaystyle\int_0^{1} \log\left(\frac{1+ax}{1-ax}\right)\frac{dx}{x\sqrt{1-x^2}} = \pi sin^{-1}a,\ a^2\le 1.$

17. Assuming that $\displaystyle\int_0^{1} x^a\, dx = \frac{1}{1+a},\ (a>-1)$; deduce that $\displaystyle\int_0^{1} \frac{x^{a-1}}{\log x}\, dx = \log 1+a.$

18. Assuming that $\displaystyle\int_0^{1} x^{a-1}\, dx = \frac{1}{a},\ (a>0)$; deduce that $\displaystyle\int_0^{1} \frac{x^{b-1}-x^{a-1}}{\log x}\, dx = \log\left(\frac{b}{a}\right),\ a>0, b>0.$

19. Show that $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{x}}{2}e^{-2|a|}$ and $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{x}}{2}e^{-2|a|}$.

20. Evaluate $\int_0^\infty e^{-xy}\cos mx\,dx$. Deduce that $\int_0^\infty e^{-x^2}\cos mx\,dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$.

21. Let $\int_0^\infty e^{-xy}\,dx = \frac{1}{y}$, $y > 0$. Deduce that $\int_0^\infty \frac{e^{-ax}-e^{-ax}}{x}\,dx = \log\left(\frac{b}{a}\right)$, $a > 0,\ b > 0$.

22. Under certain condition show that $\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = f(0)\log\left(\frac{b}{a}\right)$, $a > 0,\ b > 0$.

23. Show that $\int_0^\infty \frac{\cos xy}{1+x^2}\,dx = \frac{1}{2}\pi e^{-y}$ and $\int_0^\infty \frac{\sin xy}{x(1+x^2)}\,dx = \frac{1}{2}\pi(1-e^{-y})$, $y > 0$.

24. Starting from $\int_0^\infty e^{-\alpha x}\cos\beta x\,dx = \frac{\alpha}{\alpha^2+\beta^2}$, $\alpha \geq 0$ show that $\int_0^\infty e^{-\alpha x}\sin\left(\frac{\beta x}{x}\right)\,dx = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

25. Evaluate $\int_0^\infty e^{-tx^2}\,dx$, show that $\int_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x^2}\,dx = \sqrt{\pi}(\sqrt{b}-\sqrt{a}\,)$, where $a > 0,\ b > 0$.

26. Evaluate $\int_0^\infty e^{-ax}\sin tx\,dx$, Use the result to evaluate $\int_0^\infty e^{-ax}\,\frac{\cos bx-\cos cx}{x}\,dx$, where $a > 0$.

27. Given $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, $a > 0$, evaluate $\int_0^\infty \frac{1-e^{-ax^2}}{xe^{x^2}}\,dx$.

28. Using $\int_0^\infty \frac{dx}{1+\alpha^2x^2} = \frac{\pi}{2\alpha}$, $\alpha > 0$, show that $\int_0^\infty \frac{\tan^{-1}(bx)-\tan^{-1}(ax)}{x}\,dx = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$, $b > a > 0$.

29. Using $\int_0^\infty \frac{\sin\alpha x}{x} = \frac{\pi}{2}$, $\alpha > 0$, show that $\int_0^\infty \frac{\cos ax-\cos bx)}{x^2}\,dx = \frac{\pi}{2}(b-a)$, $b > a > 0$.

30. Using $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$, $\alpha > 0$, show that $\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$.

Vector Integration:

1. **Define the following terms:**

   (a) Vector field.

   (b) Divergence of a vector function.

   (c) Curl of a vector function.

   (d) Irrotational vector field.

   (e) Solenoidal vector field.

   (f) Independent path.

   (g) Circulation of a vector function.

   (h) Conservative Force Field.

   (i) Scalar potential.

   (j) Line Integrals.

   (k) Conservative vector field.

   (l) Work done.

2. **State the following Theorems:**

   (a) Fundamental Theorem for line integrals.

   (b) Gauess'Divergence Theorem.

   (c) Stokes'Theorem in space.

   (d) Green's Theorem in a plane.

3. Prove that if $\vec{F}$ is a continuous vector function defined in a region R, then $\oint_C \vec{F} \cdot \vec{dr}$ is independent of the path if and only if there exists a single valued scalar point function ϕ having continuous first order partial derivatives in R such that $\vec{F} = \vec{\nabla}\phi$.

4. Prove that for a continuous vector function $\vec{F}$ defined in a simply connected region R, $\oint_C \vec{F} \cdot \vec{dr}$ is independent of the path joining any two points in R if and only if $\oint_C \vec{F} \cdot \vec{dr} = 0$, for every simple closed path C in R.

5. Prove that for a continuous vector function $\vec{F}$ defined in a simply connected region R, $\oint_C \vec{F} \cdot \vec{dr} = 0$ around every simple closed path C in R , if $\vec{\nabla} \times \vec{F} = \vec{0}$ every where in R.

6. Prove the necessary and sufficient condition that $\oint_C \vec{F} \cdot \vec{dr}$ is independent of the path joining any two points in R is that $\vec{\nabla} \times \vec{F} = \vec{0}$ every where in R.

7. If $\vec{F}$ be a irrotational vector in a simply connected region R, show that there exist scalar point function ϕ such that $\vec{F} = \vec{\nabla}\phi$.

8. Define irrotational vector field. Show that the vector field $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ is irrotational. Also find the scalar function ϕ such that $\vec{F} = \vec{\nabla}\phi$.

9. Find the circulation of the vector function $\vec{F}$ around the curve $x^2+y^2 = 1, z = 0$, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

10. Find the work done in moving a particle once around the circle $x^2 + y^2 = 9$ in the xy-plane, where the vector field $\vec{F}$ is given by $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$.

11. Find the circulation of the vector function $\vec{F}$ around the curve C, where $\vec{F} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$, where C is the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$ and $x = y^2$ from $(1, 1)$ to $(0, 0)$.

12. Find the circulation of $\vec{F} = (2x-y+4z)\hat{\imath}+(x+y-z^2)\hat{\jmath}+(3x-2y+4z^2)\hat{k}$ along the circle $x^2+y^2=9, z=0$

13. Let $\vec{F} = (2y+3)\hat{\imath} + xz\hat{\jmath} + (yz-x)\hat{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the following path C.

    (a) $x = 2t^2, y = t, z = t^3$, from $t = 0$ to $t = 1$.

    (b) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(2,1,1)$.

    (c) the straight lines joining from $(0,0,0)$ to $(2,1,1)$.

14. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{\imath} + (2xz-y)\hat{\jmath} + z\hat{k}$ along the path.

    (a) $x = 2t^2, y = t, z = 4t^2 - t$, from $t = 0$ to $t = 1$.

    (b) the curve defined by $x^2 = 4y, 3x^3 = 8z$, from $x = 0$ to $x = 2$.

    (c) the straight lines joining from $(0,0,0)$ to $(2,1,3)$.

15. Prove that $\vec{F} = (y^2 \cos x + z^3)\hat{\imath} + (2y \sin x - 4)\hat{\jmath} + (3xz^2 + 2)\hat{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$. Also find the work done in moving a particle in the field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$.

16. Prove that $\vec{F} = r^2\, \vec{r}$ is a conservative force field. Find the scalar potential for $\vec{F}$.

17. Show that $\vec{F} = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}$ is a conservative force field. Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

18. Determine whether the force field $\vec{F} = 2xz\hat{\imath} + (x^2 - y)\hat{\jmath} + (2z - x^2)\hat{k}$ is a conservative force field or not.

19. Let $\vec{F} = (yz + 2x)\hat{\imath} + xz\hat{\jmath} + (2z + xy)\hat{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by $x^2 + y^2 = 1, z = 1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$.

20. Evaluate $\iint_S \vec{A} \cdot \hat{n}\; ds$, where $\vec{A} = 18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}$, where S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

21. Evaluate $\iint_S \vec{A} \cdot \hat{n}\; ds$, where $\vec{A} = z\hat{\imath} + x\hat{\jmath} - 3y^2z\hat{k}$, where S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

22. Evaluate $\iint_S (\vec{\nabla} \cdot \hat{n})\; ds$, where $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$, where S is the surface of the cube bounded by $x = 0,\ x = 1;\ \ y = 0,\ y = 1;\ \ z = 0,\ z = 1$.

23. Evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\; ds$, where $\vec{F} = y\hat{\imath} + (x - 2xz)\hat{\jmath} - xy\hat{k}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above xy-plane.

24. Calculate $\iint_S \vec{F} \cdot \hat{n}\; ds$, where $\vec{F} = x\hat{\imath} - y^2\hat{\jmath} + z^2\hat{k}$ taken over the region bounded by $x = 0, y = 0, z = 0, z = 4$ and $x^2 + y^2 = 9$

25. Calculate $\iint_S \vec{F} \cdot \hat{n}\; ds$, where $\vec{F} = 2y\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$, where S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$.

26. Calculate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 6z\hat{\imath} + (2x+y)\hat{\jmath} - x\hat{k}$, where S is the entire surface of the region bounded by the cylinder $x^2+z^2=9,\ x=0,\ y=0,\ z=0$ and $y=8$.

27. Calculate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 4xz\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}$, where S is the entire surface of the region above xy-plane bounded by the cone $x^2+y^2=z^2$ and the plane $z=4$.

28. Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$, where $\vec{F} = (x+2y)\hat{\imath} - 3z\hat{\jmath} + x\hat{k}$, where S is the surface of the plane $2x+y+2z=6$ bounded by $x=0,\ x=1;\ \ y=0,\ y=2$.

29. Evaluate $\iint_S \phi\,\hat{n}\,ds$, where $\phi = 4x+3y-2z$ and S is the surface of the plane $2x+y+2z=6$ bounded by $x=0,\ x=1;\ \ y=0,\ y=2$.

30. Evaluate $\iiint_V \left(\vec{\nabla}\cdot\vec{F}\right)\,dV$, $\vec{F} = (2x^2-3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}$ and V is the volume of the region bounded by planes $x=0,\ y=0,\ z=0$ and $2x+2y+z=4$.

31. Evaluate $\iiint_V \left(\vec{\nabla}\times\vec{F}\right)\,dV$, $\vec{F} = (2x^2-3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}$ and V is the volume of the region bounded by planes $x=0,\ y=0,\ z=0$ and $2x+2y+z=4$.

32. Evaluate $\iiint_V \vec{F}\,dV$, where $\vec{F} = 2xz\hat{\imath} - x\hat{\jmath} + y^2\hat{k}$ and V is the volume of the region bounded by the surfaces $x=0,\ y=0,\ y=6,\ z=x^2$ and $z=4$.

33. Verify Green's theorem in a plane for $\oint_C \{(x^2+xy)dx + xdy\}$, where C is the curve enclosing the region bounded by $y=x^2$ and $y=x$.

34. Verify Green's theorem in a plane for $\oint_C \{(2x-y^3)dx - xydy\}$, where C is the boundary of the region enclosed by $x^2+y^2=1$ and $x^2+y^2=9$.

35. Verify Green's theorem in a plane for $\oint_C \{(2xy-x^2)dx(x^2+y^2)dy\}$, where C is the boundary of the region enclosed by $y^2=x$ and $y=x^2$.

36. Use Green's theorem in a plane to show $\frac{1}{2}\oint_C (xdy-ydx)$ represents the area bounded by the simple closed curve C. Hence show that the area of the ellipse $x=a\cos t, y=b\sin t$ is πab.

37. Use Green's theorem in xy-plane to evaluate $\oint_C \{(y-\sin x)dx + \cos x dy\}$, where C is the triangle having vertices $(0,0), (\frac{\pi}{2},0)$ and $(\frac{\pi}{2},1)$. Also calculate it without using Green's theorem. Justify your result.

38. Verify Green's theorem for the function $\vec{F} = (3x^2-8y^2)\hat{\imath} + (4y-6xy)\hat{\jmath}$ over the region bounded by the curves $y=\sqrt{x}$ and $x=\sqrt{y}$.

39. Evaluate $\oint \vec{F}.d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ where C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0), (0,1,0)$.

40. Verify Stoke's theorem for the vector function $\vec{F} = (2x-y)\hat{\imath} - yz^2\hat{\jmath} - y^2\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$.

41. Use Stoke's theorem to prove $\oint_C \left(\vec{dr} \times \vec{F}\right) = \iint_S \left(\hat{n} \times \vec{\nabla}\right) \times \vec{F}$.

42. Verify Stoke's theorem for the vector function $\vec{A} = (y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}$, where S is the upper half surface of the cube $x = 0,\ y = 0,\ z = 0\ , x = 2,\ y = 2,\ z = 2$ above xy-plane.

43. Verify Stoke's theorem for the vector function $\vec{A} = xz\hat{\imath} - y\hat{\jmath} + x^2y\hat{k}$, where S is the surface of the region $x = 0,\ y = 0,\ z = 0\ , 2x + y + 2z = 8$, which is not included xz-plane.

44. Verify Divergence theorem for the vector function $\vec{A} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$, taken over the region bounded by $x^2 + y^2 = 4,\ z = 0$ and $z = 3$.

45. Verify Divergence theorem for the vector function $\vec{A} = 2x^2y\hat{\imath} - y^2\hat{\jmath} + 4xz^2\hat{k}$, taken over the region in the first octant bounded by $y^2 + z^2 = 9,\ z = 0$ and $x = 2$.

46. Prove that $\iint_S (\vec{A} \cdot \hat{n})\ ds = (a + b + c)V$, where S is a closed surface enclosing a volume V and the vector function $\vec{A} = ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$.

47. Let $\vec{H} = \vec{\nabla} \times \vec{A}$. Prove that $\iint_S (\vec{H} \cdot \hat{n})\ ds = 0$ for any closed surface S.

48. Let $\hat{n}$ is the outer drawn unit normal vector to any closed surface of area S. Prove that $\iiint_V \left(\vec{\nabla} \cdot \hat{n}\right)\ dV = S$.

49. Prove that (a) $\iint_S \hat{n}\ dS = 0$, for any closed surface S. (b) $\iint_S \vec{r} \times \vec{dS} = \vec{0}$, for any closed surface S.

50. A vector $\vec{A}$ is always normal to a given closed surface S. Prove that $\iiint_V \left(\vec{\nabla} \times \vec{A}\right) dV = \vec{0}$, where V is the region bounded by S.

Some problems on divergence and curl of a vector function:

1. Prove that the vector $\vec{A} = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} - 3x^2y^2\hat{k}$ is solenoidal.

2. Find the most general differentiable function $f(r)$ so that $f(r)\vec{r}$ is solenoidal.

3. For what value of the constant a, the vector $\vec{A} = (axy - z^3)\hat{\imath} + (a - 2)x^2\hat{\jmath} + (1 - a)xz^2\hat{k}$ have its curl identically equal to zero ?

4. Prove that $\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right)$.

5. Let $\vec{A}$ and $\vec{B}$ are irrotational. Prove that $\vec{A} \times \vec{B}$ is solenoidal.

6. If $f(r)$ be differentiable vector function then prove that $f(r)\vec{r}$ is irrotational.

7. Let $\vec{a}$ is a constant vector and $\vec{F} = \vec{a} \times \vec{r}$. Prove that $\vec{\nabla} \cdot \vec{F} = 0$.

8. Let u and v are differentiable scalar field. Prove that $\vec{\nabla}u \times \vec{\nabla}v$ is solenoidal.

9. Prove that $\left(\vec{U} \cdot \vec{\nabla}\right)\vec{U} = \frac{1}{2}\vec{\nabla}U^2 - \vec{U} \times \left(\vec{\nabla} \times \vec{U}\right)$.

10. Prove that $\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot \vec{\nabla}\right)\vec{A} - \vec{B}\left(\vec{\nabla} \cdot \vec{A}\right) - \left(\vec{A} \cdot \vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla} \cdot \vec{B}\right)$.